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On a formula on the potential operators of absorbing Lévy processes in the half space

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Abstract

A representation of the potential operator of an absorbing Lévy process in the half space $(0, \infty) \times \mathbb{R}^{d-1}$, $d \geq 2$, is given in terms of three measures μ, $\widehat{\mu}$ and $\dot{\mu}$ on $[0, \infty) \times \mathbb{R}^{d-1}$ arising in the fluctuation theory of Lévy processes. In the case of a rotation invariant stable Lévy process, the potential kernel in the half space is computed explicitly. It will also be proved that the measure $\widehat{\mu}$ is an excessive measure (an invariant measure under some conditions) of a Markov process, which is derived from the given Lévy process in a certain way.
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1. Introduction

The general theory of fluctuations for Lévy processes was investigated by many authors (e.g. see Bertoin [1, VI], Doney [2], Sato [7, Chapter 9] and references therein). The present paper concerns a special application, and particularly we use a result of Tamura–Tanaka [9]. Let $\{X(t)\}$ be a Lévy process in $\mathbb{R}^d$ $(d \geq 2)$ with $X(0) = 0$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$. All the sample paths are assumed to be right continuous with left limits. We write $X(t) = (X'(t), X''(t))$, $X'(t)$ being the first 1-dimensional component and $X''(t)$ the other $(d-1)$-dimensional component. A point of $\mathbb{R}^d$ is denoted by $x = (x', x'')$ with

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$x' \in \mathbb{R}$ and $x'' \in \mathbb{R}^{d-1}$. The notation $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^d$ or in $\mathbb{R}^{d-1}$. Let $\mathbb{R}^d_+ = \{x = (x', x'') : x' \geq 0, x'' \in \mathbb{R}^{d-1}\}$. We set

$$T(x) = \inf\{t > 0 : x' + X'(t) \leq 0\}, \quad x = (x', x''),$$

and consider the potential operator G of the absorbing process $\{x + X(t), t < T(x)\}$:

$$Gf(x) = E\left\{\int_0^{T(x)} f(x + X(t))\mathrm{d}t\right\}, \quad x = (x', x''),\ x' > 0.$$

In this paper, by making use of the result in [9] on the fluctuation theory for Lévy proceses, we give a representation of G in terms of three measures $\mu, \widehat{\mu}$ and $\dot{\mu}$ on $\mathbb{R}^d_+$. When $\{X(t)\}$ is a 1-dimensional symmetric stable Lévy process, our result (in particular, the formura (1.6)) goes back to Ray [6], and its extension to a wide class of 1-dimensional Lévy processes was given by Silverstein [8]. A further extension, still in the 1-dimensional case is given in Tanaka [10] (see also [11]). The present result is the multidimensional extension and can be stated as follows.

Theorem 1. *Assume that $\{X(t)\}$ is a Lévy process in $\mathbb{R}^d$ not being identically zero. Let $\mu, \widehat{\mu}$ and $\dot{\mu}$ be the measures in $\mathbb{R}^d_+$ determined by*

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{-\theta x' + \mathrm{i}\langle \xi, x\rangle} \mu(\mathrm{d}x) = \exp\left[\int_0^\infty E\left\{\mathrm{e}^{\theta X'(t) - \mathrm{i}\langle \xi, X(t)\rangle} - \mathrm{e}^{-t}; X'(t) < 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{1.1}$$

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{-\theta x' + \mathrm{i}\langle \xi, x\rangle} \widehat{\mu}(\mathrm{d}x) = \exp\left[\int_0^\infty E\left\{\mathrm{e}^{-\theta X'(t) + \mathrm{i}\langle \xi, X(t)\rangle} - \mathrm{e}^{-t}; X'(t) > 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{1.2}$$

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{\mathrm{i}\langle \xi, x\rangle} \dot{\mu}(\mathrm{d}x) = \exp\left[\int_0^\infty E\left\{\mathrm{e}^{\mathrm{i}\langle \xi'', X''(t)\rangle} - \mathrm{e}^{-t}; X'(t) = 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{1.3}$$

respectively, where $\theta > 0$ and $\xi = (\xi', \xi'') \in \mathbb{R} \times \mathbb{R}^{d-1}$. Then μ and $\widehat{\mu}$ are finite on sets of the form $[0, a] \times \mathbb{R}^{d-1}$ $(0 < a < \infty)$ and $\dot{\mu}$ is of the form $\delta_0 \otimes \mu''$, where δ_0 is the 1-dimensional δ-measure at the origin and μ'' is a finite measure on $\mathbb{R}^{d-1}$, and further

$$Gf(x) = \int_{[0,x')\times\mathbb{R}^{d-1}} \mu(\mathrm{d}u) \int_{\mathbb{R}^d_+} \widehat{\mu}(\mathrm{d}v) \int_{\mathbb{R}^d_+} \dot{\mu}(\mathrm{d}w) f(x - u + v + w) \tag{1.4}$$

holds for any $x \in (0, \infty) \times \mathbb{R}^{d-1}$ and any bounded continuous function f in $\mathbb{R}^d_+$ vanishing on $(a, \infty) \times \mathbb{R}^{d-1}$ for some $a \in (0, \infty)$. If $\{X'(t)\}$ is not a compound Poisson process, then $\dot{\mu}$ is the δ-measure at $\mathbf{0}$ (the origin in $\mathbb{R}^d$), and so

$$Gf(x) = \int_{[0,x')\times\mathbb{R}^{d-1}} \mu(\mathrm{d}u) \int_{\mathbb{R}^d_+} \widehat{\mu}(\mathrm{d}v) f(x - u + v). \tag{1.5}$$

Here are some remarks.

Remark 2. By the last statement of Theorem 1, only the two measures μ and $\widehat{\mu}$ are essential in most cases. We call μ the associated measure of $\{X(t)\}$, and $\widehat{\mu}$ the co-associated measure of $\{X(t)\}$. The associated measure of $\{X(t)\}$ is the co-associated measure of $\{-X(t)\}$. In particular, if $\{X(t)\}$ is symmetric (in the sense that $\{X(t)\}$ is identical in law to $\{-X(t)\}$, then $\mu = \widehat{\mu}$.

Remark 3. Let $\widetilde{\mu} = \widehat{\mu} * \dot{\mu}$ (convolution). Then (1.2) holds with the replacement of $\widehat{\mu}$ by $\widetilde{\mu}$ and '$X'(t) > 0$' by '$X'(t) \geq 0$', and (1.5) holds in general with $\widehat{\mu}$ replaced by $\widetilde{\mu}$. In particular, if μ

and $\widetilde{\mu}$ have densities $\varphi(x)$ and $\widetilde{\varphi}(x)$ (with respect to the Lebesgue measure), then the potential kernel $g(x, y)$, defined by $Gf(x) = \int_{\mathbb{R}_+^d} g(x, y) f(y)\,\mathrm{d}y$, admits the representation

$$g(x, y) = \int_{(0,x'\wedge y']\times\mathbb{R}^{d-1}} \varphi(x-u)\widetilde{\varphi}(y-u)\mathrm{d}u, \tag{1.6}$$

where $x' \wedge y'$ is the minimum of x' and y'.

Remark 4. Let $\widetilde{T}(x) = \inf\{t > 0 : x + X'(t) < 0\}$ and define $\widetilde{G}f$ by $\widetilde{G}f(x) = E\{\int_0^{\widetilde{T}(x)} f(x + X(t))\mathrm{d}t\}$. Then

$$\widetilde{G}f(x) = \int_{[0,x']\times\mathbb{R}^{d-1}} \mu(\mathrm{d}u) \int_{\mathbb{R}_+^d} \widehat{\mu}(\mathrm{d}v) \int_{\mathbb{R}_+^d} \dot{\mu}(\mathrm{d}w) f(x-u+v+w) \tag{1.7}$$

(notice a slight change of the integration domain of μ).

In the special case where $\{X(t)\}$ is a rotation invariant stable Lévy process, we have a more explicit representation of $g(x, y)$.

Theorem 5. *When $\{X(t)\}$ is a rotation invariant stable Lévy process on $\mathbb{R}^d$ with*

$$E\left\{\mathrm{e}^{\mathrm{i}\langle\xi, X(t)\rangle}\right\} = \mathrm{e}^{-t|\xi|^\alpha} \quad (0 < \alpha \le 2 : \textit{ constant}), \tag{1.8}$$

then the measures μ, $\widehat{\mu}$ and $\dot{\mu}$ are given by

$$\mu(\mathrm{d}x) = \pi^{-d/2}\Gamma(d/2)\Gamma(\alpha/2)^{-1}(x')^{\alpha/2}|x|^{-d}\mathrm{d}x, \tag{1.9}$$

$$\widehat{\mu} = \mu, \quad \dot{\mu} = \textit{the } \delta\textit{-measure at } \mathbf{0}, \tag{1.10}$$

and the potential kernel is represented by

$$g(x, y) = \pi^{-d}\Gamma(d/2)^2\Gamma(\alpha/2)^{-2} \int_{(0,x'\wedge y']\times\mathbb{R}^{d-1}} \frac{(x'-u')^{\alpha/2}}{|x-u|^d} \cdot \frac{(x'-u')^{\alpha/2}}{|y-u|^d}\mathrm{d}u. \tag{1.11}$$

We are also interested in the probabilistic meanings of μ and $\widehat{\mu}$. Since μ is the co-associated measure of $\{-X(t)\}$, we consider only the co-associated measure $\widehat{\mu}$ of $\{X(t)\}$. Given a Lévy process $\{X(t)\}$ we set

$$N(t) = \inf\{X'(r) : 0 \le r \le t\}$$
$$\tau(t) = \sup\{r \in [0, t] : X'(r-) \wedge X'(r) = N(t)\} \quad (X'(0-) = 0),$$
$$X(\tau(t)^*) = \begin{cases} X(\tau(t)) & \text{if } X'(\tau(t)) = N(t), \\ X(\tau(t)-) & \text{if } X'(\tau(t)) \ne N(t), \end{cases}$$
$$c = \exp\left\{-\int_0^\infty (1-\mathrm{e}^{-t})P(X'(t) \le 0)\frac{\mathrm{d}t}{t}\right\}.$$

Note that $X'(\tau(t)^*) = N(t)$. We define a process $Y(t; x)$ starting from $x \in \mathbb{R}_+^d$ by

$$Y(t, x) = \begin{cases} x + X(t) & \text{if } x' + N(t) > 0, \\ X(t) - X(\tau(t)^*) & \text{if } x' + N(t) \le 0. \end{cases} \tag{1.12}$$

Theorem 6. (i) *$\mathbb{Y} = \{Y(t; x), t \geq 0; x \in \mathbb{R}^d_+\}$ is a Markov process on $\mathbb{R}^d_+$.*
(ii) *If $c > 0$, then $\widehat{\mu}$ is an excessive measure for $\mathbb{Y}$. More precisely*

$$\widehat{\mu}(A) = cE\left\{\int_0^\infty \mathbf{1}_A(Y(t; \mathbf{0}))\mathrm{d}x\right\} \tag{1.13}$$

$$\int_{\mathbb{R}^d_+} \widehat{\mu}(\mathrm{d}x)Q(t, x, A) = \widehat{\mu}(A) - c\int_0^t Q(s, \mathbf{0}, A)\mathrm{d}s, \tag{1.14}$$

where $\mathbf{1}_A$ is the indicator function of A and $Q(t, x, A)$ is the transition function of $\mathbb{Y}$.
(iii) *If $c = 0$, $\widehat{\mu}$ is an invariant measure for $\mathbb{Y}$.*

The rest of this paper is divided into three sections, in which the above theorems will be proved.

2. The measures μ, $\widehat{\mu}$ and $\dot{\mu}$, and the potential operator G

2.1

We prepare some consequences of the result of [9]. We set

$$\sigma(t) = \inf\{r \in [0, t] : X'(r-) \wedge X'(r) = N(t)\},$$
$$X(\sigma(t)^*) = \begin{cases} X(\sigma(t)) & \text{if } X'(\sigma(t)) = N(t), \\ X(\sigma(t)-) & \text{if } X'(\sigma(t)) \neq N(t). \end{cases}$$

Note that $X'(\sigma(t)^*) = X'(\tau(t)^*) = N(t)$. Let $\lambda > 0$ and consider an exponential random variable ζ_λ with mean $1/\lambda$. Enlarging the probability space if necessary, we assume that ζ_λ is defined on $(\Omega, \mathcal{F}, P)$ and that ζ_λ is independent of $\{X(t)\}$. Since $X(\tau(t)^*)$ is right continuous, $X(\tau(\zeta_\lambda)^*)$ is $\mathcal{F}$-measurable. Taking into account of the completeness assumption of $(\Omega, \mathcal{F}, P)$, we can also prove that $X(\sigma(\zeta_\lambda)^*)$ is $\mathcal{F}$-measurable. We now rewrite Corollary of [9] in terms of the Lévy process $\{-X(t)\}$ (instead of $\{X(t)\}$) and then set $\alpha = \beta = \gamma = 0$. As a result we obtain the following lemma.

Lemma 7. (i) *The random variables $X(\sigma(\zeta_\lambda)^*)$, $X(\zeta_\lambda) - X(\tau(\zeta_\lambda)^*)$ and $X(\tau(\zeta_\lambda)^*) - X(\sigma(\zeta_\lambda)^*)$ are independent.*
(ii) *Let $\theta \geq 0$ and $\xi = (\xi', \xi'') \in \mathbb{R} \times \mathbb{R}^{d-1}$. Then*

$$E\left\{\mathrm{e}^{\theta N(\zeta_\lambda) - \mathrm{i}\langle \xi, X(\sigma(\zeta_\lambda)^*)\rangle}\right\}$$
$$= \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{\theta X'(t) - \mathrm{i}\langle \xi, X(t)\rangle} - 1; X'(t) < 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{2.1}$$

$$E\left\{\mathrm{e}^{-\theta(X'(\zeta_\lambda) - N(\zeta_\lambda)) - \mathrm{i}\langle \xi, X(\zeta_\lambda) - X(\tau(\zeta_\lambda)^*)\rangle}\right\}$$
$$= \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{-\theta X'(t) - \mathrm{i}\langle \xi, X(t)\rangle} - 1; X'(t) > 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{2.2}$$

$$E\left\{\mathrm{e}^{\mathrm{i}\langle \xi, X(\tau(\zeta_\lambda)^*) - X(\sigma(\zeta_\lambda)^*)\rangle}\right\}$$
$$= \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{\mathrm{i}\langle \xi'', X''(t)\rangle} - 1; X'(t) = 0\right\} \frac{\mathrm{d}t}{t}\right]. \tag{2.3}$$

For each fixed $\lambda > 0$, let $\nu_\lambda, \widehat{\nu}_\lambda$ and $\dot{\nu}_\lambda$ be the probability distributions of $-X(\sigma(\zeta_\lambda)^*)$, $X(\zeta_\lambda) - X(\tau(\zeta_\lambda)^*)$ and $X(\tau(\zeta_\lambda)^*) - X(\sigma(\zeta_\lambda)^*)$, respectively. Since the first component of $X(\tau(\zeta_\lambda)^*) - X(\sigma(\zeta_\lambda)^*)$ is identically zero, $\dot{\nu}_\lambda = \delta_0 \otimes \dot{\nu}''_\lambda$ where $\dot{\nu}''_\lambda$ is the probability distribution of $X''(\tau(\zeta_\lambda)^*) - X''(\sigma(\zeta_\lambda)^*)$. With the notation $u = (u', u'') \in \mathbb{R} \times \mathbb{R}^{d-1}$ we have from (2.1)–(2.3)

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{-\theta u' + \mathrm{i}\langle \xi, u\rangle} \nu_\lambda(\mathrm{d}u) = \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{\theta X'(t) - \mathrm{i}\langle \xi, X(t)\rangle} - 1;\, X'(t) < 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{2.4}$$

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{-\theta u' + \mathrm{i}\langle \xi, u\rangle} \widehat{\nu}_\lambda(\mathrm{d}u) = \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{-\theta X'(t) + \mathrm{i}\langle \xi, X(t)\rangle} - 1;\, X'(t) > 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{2.5}$$

$$\begin{aligned}\int_{\mathbb{R}^d_+} \mathrm{e}^{\mathrm{i}\langle \xi, u\rangle} \dot{\nu}_\lambda(\mathrm{d}u) &= \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}\langle \xi'', u''\rangle} \dot{\nu}''_\lambda(\mathrm{d}u'') \\ &= \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{\mathrm{i}\langle \xi'', X''(t)\rangle} - 1;\, X'(t) = 0\right\} \frac{\mathrm{d}t}{t}\right],\end{aligned} \tag{2.6}$$

where $\theta \geq 0$ and $\xi = (\xi', \xi'') \in \mathbb{R} \times \mathbb{R}^{d-1}$.

2.2

We introduce the measures $\mu_\lambda, \widehat{\mu}_\lambda$ and $\dot{\mu}_\lambda$ with a slight modification of (2.4)–(2.6), namely by

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{-\theta u' + \mathrm{i}\langle \xi, u\rangle} \mu_\lambda(\mathrm{d}u) = \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{\theta X'(t) - \mathrm{i}\langle \xi, X(t)\rangle} - \mathrm{e}^{-t};\, X'(t) < 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{2.7}$$

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{-\theta u' + \mathrm{i}\langle \xi, u\rangle} \widehat{\mu}_\lambda(\mathrm{d}u) = \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{-\theta X'(t) + \mathrm{i}\langle \xi, X(t)\rangle} - \mathrm{e}^{-t};\, X'(t) > 0\right\} \frac{\mathrm{d}t}{t}\right], \tag{2.8}$$

$$\int_{\mathbb{R}^d_+} \mathrm{e}^{\mathrm{i}\langle \xi, u\rangle} \dot{\mu}_\lambda(\mathrm{d}u) = \exp\left[\int_0^\infty \mathrm{e}^{-\lambda t} E\left\{\mathrm{e}^{\mathrm{i}\langle \xi'', X''(t)\rangle} - \mathrm{e}^{-t};\, X'(t) = 0\right\} \frac{\mathrm{d}t}{t}\right]. \tag{2.9}$$

If we set

$$c_\lambda = \exp\left\{-\int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t}) P(X'(t) < 0) \frac{\mathrm{d}t}{t}\right\}, \tag{2.10}$$

$$\widehat{c}_\lambda = \exp\left\{-\int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t}) P(X'(t) > 0) \frac{\mathrm{d}t}{t}\right\}, \tag{2.11}$$

$$\dot{c}_\lambda = \exp\left\{-\int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t}) P(X'(t) = 0) \frac{\mathrm{d}t}{t}\right\}, \tag{2.12}$$

then

$$\nu_\lambda = c_\lambda \mu_\lambda, \quad \widehat{\nu}_\lambda = \widehat{c}_\lambda \widehat{\mu}_\lambda, \quad \dot{\nu}_\lambda = \dot{c}_\lambda \dot{\mu}_\lambda, \tag{2.13}$$

$$c_\lambda \widehat{c}_\lambda \dot{c}_\lambda = \exp\left\{-\int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t}) \frac{\mathrm{d}t}{t}\right\} = \frac{\lambda}{1+\lambda} \sim \lambda \quad \text{as } \lambda \downarrow 0. \tag{2.14}$$

Lemma 8. *Let $\{X(t), t \geq 0\}$ be a fixed Lévy process in $\mathbb{R}^d$ with Lévy measure J.*

(i) *There exist constants* K_0 *and* K_1 *such that*

$$E\left\{\left|1-\mathrm{e}^{\mathrm{i}\langle\xi,X(t)\rangle}\right|\right\} \le K_0 t + K_1|\xi|t^{\frac{1}{2}} \tag{2.15}$$

for any $\xi \in \mathbb{R}^d$ *and* $t \in [0,1]$, *where* $K_0 = 2J(|x| \ge 1)$ *and* K_1 *depends neither on* ξ *nor on* t.

(ii) *There exists a constant* K_2, *depending neither on* θ *nor on* t, *such that*

$$E\left\{1-\mathrm{e}^{-\theta|X'(t)|}\right\} \le K_0 t + K_2\theta t^{\frac{1}{2}} \tag{2.16}$$

for any $\theta \ge 0$ *and* $t \in [0,1]$, K_0 *being the same as in the case* (*i*).

(iii) *If* $\{X'(t)\}$ *is not the zero process, then for any* $\theta > 0$

$$E\left\{\mathrm{e}^{-\theta|X'(t)|}\right\} \le K_3 t^{-\frac{1}{4}} \quad \textit{for } \forall t > 0, \tag{2.17}$$

where K_3 *depends on* θ *but not on* t.

Proof. We first prove (i). We can decompose $X(t)$ as $X(t) = X_0(t) + X_1(t)$, where $\{X_0(t)\}$ is a Lévy process, $\{X_1(t)\}$ is a compound Poisson process independent of $\{X_0(t)\}$, and the Lévy measure of $\{X_1(t)\}$ is the restriction of J on $\{|x| \ge 1\}$. Since the Lévy measure of $\{X_0(t)\}$ is the restriction of J to $\{0 < |x| < 1\}$, its support is bounded. Therefore any absolute moment of positive order is finite (e.g. see [7, p.159]). Let T be the first jumping time of $\{X_1(t)\}$. Then $P\{T > t\} = \mathrm{e}^{-J(|x|\ge 1)t},\ t > 0$. Therefore

$$\begin{aligned}
E\left\{\left|1-\mathrm{e}^{\mathrm{i}\langle\xi,X(t)\rangle}\right|\right\} &= E\left\{\left|1-\mathrm{e}^{\mathrm{i}\langle\xi,X_0(t)\rangle}\right|; T > t\right\} + E\left\{\left|1-\mathrm{e}^{\mathrm{i}\langle\xi,X(t)\rangle}\right|; T \le t\right\} \\
&\le E\left\{\left|1-\mathrm{e}^{\mathrm{i}\langle\xi,X_0(t)\rangle}\right|\right\} + 2P\{T \le t\} \\
&\le |\xi|E\{|X_0(t)|\} + 2(1-\mathrm{e}^{-J(|x|\ge 1)t}) \\
&\le \text{const. } |\xi|t^{1/2} + K_0 t, \quad (0 \le \forall t \le 1)
\end{aligned}$$

which proves (i). The proof of (ii) is similar to the above. The proof of (iii) was given in [11, Lemma 2] by making use of the Berry–Esseen Theorem. □

We exclude the trivial case where $X'(t)$ is the zero process. By (iii) of Lemma 8, we have

$$\int_0^\infty (1-\mathrm{e}^{-t})P\{X'(t)=0\}\frac{\mathrm{d}t}{t} < \infty, \tag{2.18}$$

and hence

$$\dot{c}_\lambda \to \dot{c}_0 = \exp\left\{-\int_0^\infty (1-\mathrm{e}^{-t})P(X'(t)=0)\frac{\mathrm{d}t}{t}\right\} > 0 \quad \text{as } \lambda \downarrow 0. \tag{2.19}$$

It is also known that if $\{X'(t)\}$ is not a compound Poisson process, then $\int_0^\infty P\{X'(t)=0\}\mathrm{d}t = 0$ (e.g. see [7, p.372]). Therefore

$$\dot{c}_0 = 1 \quad \text{if } \{X'(t)\} \text{ is not a compound Poisson process.} \tag{2.20}$$

We are now going to let $\lambda \downarrow 0$ in (2.7)–(2.9), and then we can prove the existence of the measures μ, $\widehat{\mu}$ and $\dot{\mu}$ satisfying (1.1)–(1.3). Let $\theta > 0$ be fixed and set $\mu_{\lambda,\theta}(\mathrm{d}u) = \mathrm{e}^{-\theta u'}\mu_\lambda(\mathrm{d}u)$. By (2.7), the Fourier transform $\mathcal{F}_{\lambda,\theta}(\xi)$ of $\mu_{\lambda,\theta}$ is expressed as

$$\mathcal{F}_{\lambda,\theta}(\xi) = \int_{\mathbb{R}^d_+} \mathrm{e}^{\mathrm{i}\langle\xi,u\rangle}\mu_{\lambda,\theta}(\mathrm{d}u) = \int_0^\infty \mathrm{e}^{-\lambda t}H_\theta(\xi,t)\frac{\mathrm{d}t}{t},$$

where

$$H_\theta(\xi,t) = E\left\{e^{\theta X'(t)-i\langle\xi,X(t)\rangle} - e^{-t};\ X'(t)<0\right\}.$$

We first consider $\int_0^1 e^{-\lambda t} H_\theta(\xi,t)\frac{dt}{t}$. We write

$$\begin{aligned} H_\theta(\xi,t) &= E\left\{e^{\theta X'(t)}(e^{-i\langle\xi,X(t)\rangle}-1);\ X'(t)<0\right\} \\ &\quad + E\left\{e^{\theta X'(t)}-1;\ X'(t)<0\right\} + E\left\{1-e^{-t};\ X'(t)<0\right\} \\ &= H_\theta^{(1)}(\xi,t) + H_\theta^{(2)}(\xi,t) + H_\theta^{(3)}(\xi,t). \end{aligned}$$

Then by Lemma 8, we have

$$\begin{aligned} |H_\theta^{(1)}(\xi,t)| &\le E\left\{|1-e^{-i\langle\xi,X(t)\rangle}|\right\} \le K_0 t + K_1|\xi|t^{1/2}, \\ |H_\theta^{(2)}(\xi,t)| &\le E\left\{1-e^{-\theta|X'(t)|}\right\} \le K_0 t + K_2|\xi|t^{1/2}, \\ |H_\theta^{(3)}(\xi,t)| &\le 1-e^{-t} \le t. \end{aligned}$$

Therefore $\int_0^1 |H_\theta(\xi,t)|\frac{dt}{t} < \infty$, and for each $\lambda \ge 0$ $\int_0^1 e^{-\lambda t} H_\theta(\xi,t)\frac{dt}{t}$ is continuous in ξ. Similarly $\int_1^\infty |H_\theta(\xi,t)|\frac{dt}{t} < \infty$ and for each $\lambda \ge 0$ $\int_1^\infty e^{-\lambda t} H_\theta(\xi,t)\frac{dt}{t}$ is continuous in ξ. Therefore, for each $\lambda \ge 0$ $\mathcal{F}_{\lambda,\theta}(\xi)$ is continuous in ξ and

$$\lim_{\lambda\downarrow 0} \mathcal{F}_{\lambda,\theta}(\xi) = \mathcal{F}_{0,\theta}(\xi) \quad \text{for all } \xi \in \mathbb{R}^d.$$

Thus, there exists a finite measure $\mu_{0,\theta}$ on $\mathbb{R}_+^d$ such that for any $f \in C_0(\mathbb{R}_+^d)$, the space of continuous functions on $\mathbb{R}_+^d$ with compact supports

$$\lim_{\lambda\downarrow 0}\int_{\mathbb{R}_+^d} f(u)\mu_{\lambda,\theta}(du) = \int_{\mathbb{R}_+^d} f(u)\mu_{0,\theta}(du).$$

Let $\mu = e^{\theta u'}\mu_{0,\theta}(du)$. Then μ is a measure on $\mathbb{R}_+^d$, finite on sets of the form $[0,a]\times\mathbb{R}^{d-1}$ $(0 < a < \infty)$, such that (1.1) holds for $\theta > 0$ and

$$\lim_{\lambda\downarrow 0}\int_{\mathbb{R}_+^d} f(u)\mu_\lambda(du) = \int_{\mathbb{R}_+^d} f(u)\mu(du), \quad f \in C_0(\mathbb{R}_+^d).$$

This equation also implies that μ is independent of θ.

Similarly, there exists a measure $\widehat{\mu}$ on $\mathbb{R}_+^d$, finite on sets of the form $[0,a]\times\mathbb{R}^{d-1}$ $(0 < a < \infty)$, such that (1.2) holds for $\theta > 0$ and

$$\lim_{\lambda\downarrow 0}\int_{\mathbb{R}_+^d} f(u)\widehat{\mu}_\lambda(du) = \int_{\mathbb{R}_+^d} f(u)\widehat{\mu}(du), \quad f \in C_0(\mathbb{R}_+^d).$$

We can also prove that there exists a finite measure $\dot{\mu}''$ on $\mathbb{R}^{d-1}$, with $\dot{\mu}''(\mathbb{R}^{d-1}) = \dot{c}_0^{-1} < \infty$, such that (1.3) holds for $\dot{\mu} = \delta_0 \otimes \dot{\mu}''$ and

$$\lim_{\lambda\downarrow 0}\int_{\mathbb{R}^{d-1}} g(u)\dot{\mu}_\lambda''(du'') = \int_{\mathbb{R}_+^d} g(u)\dot{\mu}''(du''), \quad f \in C_0(\mathbb{R}_+^d)$$

for any bounded continuous function g on $\mathbb{R}^{d-1}$.

2.3

We give the proof of Theorem 1. Making use of the independence of $X(\sigma(\zeta_\lambda)^*)$, $X(\zeta_\lambda) - X(\tau(\zeta_\lambda)^*)$ and $X(\tau(\zeta_\lambda)^*) - X(\sigma(\zeta_\lambda)^*)$, we have for $f \in C_0(\mathbb{R}^d_+)$

$$\begin{aligned} G_\lambda f(x) &:= \int_0^\infty e^{-\lambda t} E\left\{f(x+X(t)); x'+N(t)>0\right\} dt \\ &= \lambda^{-1} E\left\{f(x+X(\zeta_\lambda)); -N(\zeta_\lambda) < x'\right\} \\ &= \lambda^{-1} E\{f[x+X(\sigma(\zeta_\lambda)^*) + X(\zeta_\lambda) - X(\tau(\zeta_\lambda)^*) + X(\tau(\zeta_\lambda)^*) \\ &\quad - X(\sigma(\zeta_\lambda)^*)]; -X'(\sigma(\zeta_\lambda)^*) < x'\} \\ &= \lambda^{-1} \int_{[0,x')\times\mathbb{R}^{d-1}} \nu_\lambda(du) \int_{\mathbb{R}^d_+} \widehat{\nu}_\lambda(dv) \int_{\mathbb{R}^d_+} \dot{\nu}_\lambda(dw) f(x-u+v+w). \end{aligned} \tag{2.21}$$

It is easy to see that

$$Gf(x) = \int_0^\infty E\{f(x+X(t)); x'+N(t)>0\} dt, \tag{2.22}$$

and hence $G_\lambda f(x) \to Gf(x)$ as $\lambda \downarrow 0$. We now let $\lambda \downarrow 0$ in (2.21). Since $\nu_\lambda \otimes \widehat{\nu}_\lambda \otimes \dot{\nu}_\lambda = c_\lambda \widehat{c}_\lambda \dot{c}_\lambda \mu_\lambda \otimes \widehat{\mu}_\lambda \otimes \dot{\mu}_\lambda$ and $\lambda^{-1} c_\lambda \widehat{c}_\lambda \dot{c}_\lambda \to 1$ as $\lambda \downarrow 0$, we obtain (1.4) holding for any $x = (x', x'')$ with $x' \in A = \{u'; \mu(\{u'\} \times \mathbb{R}^{d-1}) = 0\}$. Moreover, if $0 < x'_1 < x'_2 < \dots$ with $x'_n \to x'$ $(n \to \infty)$ and if $x_n = (x'_n, x'')$, then $Gf(x_n) \to Gf(x)$ $(n \to \infty)$ by (2.22). This property of $Gf(x)$ (so to speak, the 'left continuity') combined with the denseness of A in $(0,\infty)$ implies that (1.4) holds for all $x = (x', x'') \in (0,\infty) \times \mathbb{R}^{d-1}$. Finally, if $\{X'(t)\}$ is not a compound Poisson process, then $\int_0^\infty P\{X'(t) = 0\} dt = 0$, and so $\dot{\mu}$ is the δ-measure at $\mathbf{0}$. This implies (1.5). The proof of Theorem 1 is finished. $\square$

2.4

Before closing this section, we derive the density version of (1.6) from

$$Gf(x) = \int_{[0,x')\times\mathbb{R}^{d-1}} \mu(du) \int_{\mathbb{R}^d_+} \widetilde{\mu}(dv) f(x-u+v)$$

assuming that μ and $\widetilde{\mu}$ have densities $\varphi(x)$ and $\widetilde{\varphi}(x)$. By making use of the change of variable $x-u+v=y$ with u fixed, we have

$$Gf(x) = \int_{\mathbb{R}^d_+} \varphi(u) \int_{\mathbb{R}^d_+} \widetilde{\varphi}(y-(x-u)) f(y) \mathbf{1}_{[0,x')\times\mathbb{R}^{d-1}}(u) \mathbf{1}_{[x'-u',\infty)\times\mathbb{R}^{d-1}}(y) du\, dy,$$

and again making use of the change of variable $x-u=\xi$ the equality continues to

$$\begin{aligned} &= \int_{\mathbb{R}^d_+} \varphi(x-\xi) \int_{\mathbb{R}^d_+} \widetilde{\varphi}(y-\xi) f(y) \mathbf{1}_{(0,x']\times\mathbb{R}^{d-1}}(\xi) \mathbf{1}_{[\xi',\infty)\times\mathbb{R}^{d-1}}(y) d\xi\, dy \\ &= \int_{\mathbb{R}^d_+} \varphi(x-\xi) \int_{\mathbb{R}^d_+} \widetilde{\varphi}(y-\xi) f(y) \mathbf{1}_{(0,x']\times\mathbb{R}^{d-1}}(\xi) \mathbf{1}_{(0,y']\times\mathbb{R}^{d-1}}(\xi) d\xi\, dy \\ &= \int_{\mathbb{R}^d_+} \left\{ \int_{(0,x'\wedge y']\times\mathbb{R}^{d-1}} \varphi(x-\xi)\widetilde{\varphi}(y-\xi) d\xi \right\} f(y) dy. \end{aligned}$$

The last expression implies (1.6).

3. Rotation invariant stable Lévy processes

We give a proof of Theorem 5. Let $\{X(t)\}$ be a rotation invariant stable Lévy process with (1.8). We are going to find the explicit form of the measure μ on $\mathbb{R}_+^d$ determined by

$$\int_{\mathbb{R}_+^d} \mathrm{e}^{-\theta x' + \mathrm{i}\langle \xi'', x''\rangle} \mu(\mathrm{d}x'\mathrm{d}x'') = \exp\left[\int_0^\infty E\left\{\mathrm{e}^{\theta X'(t) - \mathrm{i}\langle \xi'', X''(t)\rangle} - \mathrm{e}^{-t}; X'(t) < 0\right\}\frac{\mathrm{d}t}{t}\right], \tag{3.1}$$

where $\theta > 0$ and $\xi'' \in \mathbb{R}^{d-1}$. Since $(-X'(t), X''(t))$ is equivalent in law to $X(t)$, we have

$$\begin{aligned} E\{\mathrm{e}^{\theta X'(t) - \mathrm{i}\langle \xi'', X''(t)\rangle} - \mathrm{e}^{-t}; X'(t) < 0\} &= E\{\mathrm{e}^{-\theta X'(t) - \mathrm{i}\langle \xi'', X''(t)\rangle} - \mathrm{e}^{-t}; X'(t) > 0\} \\ &= \frac{1}{2} E\left\{\mathrm{e}^{-\theta |X'(t)| - \mathrm{i}\langle \xi'', X''(t)\rangle} - \mathrm{e}^{-t}\right\}. \end{aligned}$$

Making use of the fact that $\mathrm{e}^{-\theta|z|}$ is the characteristic function of the 1-dimensional Cauchy distribution, we have with $\widetilde{\xi} = (\xi', -\xi'')$

$$\begin{aligned} E\left\{\mathrm{e}^{-\theta |X'(t)| - \mathrm{i}\langle \xi'', X''(t)\rangle} - \mathrm{e}^{-t}\right\} &= \frac{1}{\pi}\int_{-\infty}^\infty E\{\mathrm{e}^{\mathrm{i}\langle \widetilde{\xi}, X(t)\rangle} - \mathrm{e}^{-t}\}\frac{\theta}{\theta^2 + (\xi')^2}\mathrm{d}\xi' \\ &= \frac{1}{\pi}\int_{-\infty}^\infty (\mathrm{e}^{-t|\xi|^\alpha} - \mathrm{e}^{-t})\frac{\theta}{\theta^2 + (\xi')^2}\mathrm{d}\xi' \end{aligned}$$

and so

$$\begin{aligned} &\log\{\text{the right-hand side of (3.1)}\} \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \left\{\int_0^\infty \frac{\mathrm{e}^{-t|\xi|^\alpha} - \mathrm{e}^{-t}}{t}\mathrm{d}t\right\}\frac{\theta}{\theta^2 + (\xi')^2}\mathrm{d}\xi' = -\frac{\alpha\theta}{4\pi}\int_{-\infty}^\infty \frac{\log|\xi|^2}{\theta^2 + (\xi')^2}\mathrm{d}\xi'. \end{aligned}$$

Using $\int_0^\infty (b^2 + x^2)^{-1}\log(a^2 + x^2)\mathrm{d}x = \frac{\pi}{b}\log(a + b)$ $(a > 0, b > 0)$, we have

$$\log\{\text{the right-hand side of (3.1)}\} = \log\left\{(\theta + |\xi''|)^{-\alpha/2}\right\}. \tag{3.2}$$

Now assuming that $\mu(\mathrm{d}x) = \varphi(x', x'')\mathrm{d}x'\mathrm{d}x''$, we set

$$\varphi_{\xi''}(x') = \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}\langle \xi'', x''\rangle}\varphi(x', x'')\mathrm{d}x''.$$

Then (3.1) and (3.2) yield

$$\int_0^\infty \mathrm{e}^{-\theta x'}\varphi_{\xi''}(x')\mathrm{d}x' = (\theta + |\xi''|)^{-\alpha/2}.$$

Using the formula $\int_0^\infty \mathrm{e}^{-\lambda t}\gamma_\alpha(t)\mathrm{d}t = (\lambda + 1)^{-\alpha}$, where $\gamma_\alpha(t) = \Gamma(\alpha)^{-1}t^{\alpha-1}\mathrm{e}^{-t}$, we have

$$\varphi_{\xi''}(x') = |\xi''|^{1-\alpha/2}\gamma_{\alpha/2}(|\xi''|x') = \Gamma(\alpha/2)^{-1}(x')^{\frac{\alpha}{2}-1}\mathrm{e}^{-|\xi''|x'},$$

and hence

$$\int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}\langle \xi'', x''\rangle}\varphi(x', x'')\mathrm{d}x'' = \Gamma(\alpha/2)^{-1}(x')^{\frac{\alpha}{2}-1}\mathrm{e}^{-x'|\xi''|}.$$

Noting that $e^{-c|\xi''|}$ is the characteristic function of the $(d-1)$-dimensional Cauchy density $\pi^{-d/2}\Gamma(d/2)c(c^2+|x''|^2)^{-d/2}$, we finally obtain

$$\varphi(x', x'') = \pi^{-d/2}\Gamma(d/2)\Gamma(\alpha/2)^{-1}(x')^{\alpha/2}|x|^{-d},$$

from which the assertion of Theorem 5 follows.

4. Probabilistic meaning of $\widehat{\mu}$

For $a > 0$ we set

$$\begin{aligned} X_s(t) &= X(s+t) - X(s), \quad t \geq 0, \\ N_s(t) &= \inf\{X'_s(r) : 0 \leq r \leq t\}, \\ \tau_s(s) &= \sup\{r \in [0,t] : X'_s(r-) \wedge X'_s(r) = N_s(t)\}, \end{aligned}$$

and let $X_s(\tau_s(t)^*) = X_s(\tau_s(t))$ or $= X_s(\tau_s(t)-)$ according to $X_s(\tau_s(t)) = N_s(t)$ or to $X'(\tau_s(t)) \neq N_s(t)$.

Lemma 9. *Let $x \in \mathbb{R}^d_+$, $s \geq 0$ and $t > 0$. Then*

$$Y(s+t;x) = \begin{cases} Y(s;x) + X_s(t) & \text{if } Y'(s;x) + N_s(t) > 0, \\ X_s(t) - X_s(\tau_s(t)^*) & \text{if } Y'(s;x) + N_s(t) \leq 0. \end{cases} \tag{4.1}$$

The proof of this lemma can be given by considering the following four cases separately.

(a) $Y'(s;x) + N_s(t) > 0,\ x' + N(s) > 0$,
(b) $Y'(s;x) + N_s(t) > 0,\ x' + N(s) \leq 0$,
(c) $Y'(s;x) + N_s(t) \leq 0,\ x' + N(s) > 0$,
(d) $Y'(s;x) + N_s(t) \leq 0,\ x' + N(s) \leq 0$.

For example, the proof in the case (c) is as follows. From the second condition of (c) $Y(s;x) = x + X(s)$. Therefore

$$\begin{aligned} \text{(c)} &\Leftrightarrow x' + X'(s) + \inf_{0\leq r\leq t}\left\{X'(s+r) - X'(s)\right\} \leq 0, \quad x' + N(s) > 0, \\ &\Leftrightarrow x' + \inf_{0\leq r\leq t} X'(s+r) \leq 0, \quad x' + N(s) > 0. \end{aligned}$$

Thus, under the condition (c)

$$x' + N(s+t) \leq 0 < x' + N(s). \tag{4.2}$$

The first inequality of the above implies

$$Y(s+t;x) = X(s+t) - X(\tau(s+t)^*).$$

So it is enough to show that

$$X(s+t) - X(\tau(s+t)^*) = X_s(t) - X_s(\tau_s(t)^*). \tag{4.3}$$

The right-hand side of the above is equal to

$$X(s+t) - X(s) - X_s(\tau_s(t)^*).$$

For the proof of (4.3), it is enough to show

$$X(s) + X_s(\tau_s(t)^*) = X(\tau(s+t)^*). \tag{4.4}$$

On the other hand, from (4.2) we have $N(s) > N(s+t)$, and this implies

$$s < \tau(s+t), \quad s + \tau_s(t) = \tau(s+t). \tag{4.5}$$

From (4.5), it is obvious that (4.4) holds (drawing a picture of $X'(\cdot)$ is helpful). The other cases can be treated similarly. □

Proof of Theorem 6. First we consider (i). From the definition (1.12) of $Y(t;x)$, it is not so hard to prove that $Y(t;x)$ is a right continuous function of t with left limits. Let $\{\mathcal{F}_t\}$ be the right continuous modification of the filtration generated by $\{X(t)\}$. Then the process $\{Y(t;x), t \geq 0\}$ is $\{\mathcal{F}_t\}$-adapted. For a bounded Borel function f on $\mathbb{R}^d_+$, we set $Q_t f(y) = E\{f(Y(t;y))\}$. Making use of Lemma 9 we have

$$\begin{aligned} E\{f(Y(s+t;x))|\mathcal{F}_s\} = {} & E\{f(Y(s;x) + X_s(t))\mathbf{1}_{(0,\infty)}(Y'(s;x) + N_s(t))|\mathcal{F}_s\} \\ & + E\{f(X_s(t)) - X_s(\tau_s(t)^*)\mathbf{1}_{(-\infty,0]}(Y'(s;x) + N_s(t))|\mathcal{F}_s\}. \end{aligned} \tag{4.6}$$

On the other hand

$$\begin{aligned} & E\{f(y + X_s(t))\mathbf{1}_{(0,\infty)}(y' + N_s(t))\} + E\{f(X_s(t) - X_s(\tau_s(t)^*))\mathbf{1}_{(-\infty,0]}(y' + N_s(t))\} \\ & \quad = E\{f(y + X(t))\mathbf{1}_{(0,\infty)}(y' + N(t))\} + E\{f(X(t) - X(\tau(t)^*))\mathbf{1}_{(-\infty,0]}(y' + N(t))\} \\ & \quad = E\{f(Y(t;y))\} = Q_t f(y). \end{aligned}$$

This, combined with the fact that $\mathcal{F}_s$ and the random vector $\{X_s(t), N_s(t), X_s(\tau_s(t)^*)\}$ are independent, implies that (4.6) is equal to $Q_t f(Y(s;x))$ a.s. Therefore, $\mathbb{Y}$ is a Markov process with transition function $Q(s,x,A) = P\{Y(s;x) \in A\}$.

The proof of (ii) of Theorem 6 is as follows. Since $c > 0$, $\lim_{t\to\infty} X'(t) = \infty$ a.s. (see [7, p.363]). From (2.11) and the identity $\int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t})\mathrm{d}t/t = \log \frac{\lambda+1}{\lambda}$, we have

$$\begin{aligned} \lim_{\lambda\downarrow 0} \widehat{c}_\lambda^{-1}\lambda &= \lim_{\lambda\downarrow 0} \exp\left\{\int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t})P(X'(t) > 0)\frac{\mathrm{d}t}{t} - \int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t})\frac{\mathrm{d}t}{t}\right\} \\ &= \lim_{\lambda\downarrow 0} \exp\left\{-\int_0^\infty \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-t})P(X'(t) \leq 0)\frac{\mathrm{d}t}{t}\right\} = c. \end{aligned}$$

Then for any $f \in C_0(\mathbb{R}^d_+)$, we have

$$\begin{aligned} \int f\mathrm{d}\widehat{\mu} &= \lim_{\lambda\downarrow 0}\int f\mathrm{d}\widehat{\mu}_\lambda = \lim_{\lambda\downarrow 0}\widehat{c}_\lambda^{-1}\int f\mathrm{d}\widehat{\nu}_\lambda = \lim_{\lambda\downarrow 0}\widehat{c}_\lambda^{-1}E\{f(X(\zeta_\lambda) - X(\tau(\zeta_\lambda)^*))\} \\ &= \lim_{\lambda\downarrow 0}\widehat{c}_\lambda^{-1}\lambda E\left\{\int_0^\infty \mathrm{e}^{-\lambda t}f(X(t) - X(\tau(t)^*))\mathrm{d}t\right\} = cE\left\{\int_0^\infty f(Y(t;\mathbf{0}))\mathrm{d}t\right\}, \end{aligned}$$

which proves (1.13). The Eq. (1.14) is proved easily.

We proceed to the proof of (iii), so assume $c = 0$. When $X(t)$ is decreasing, (iii) is trivial. So we assume that $X(t)$ is not decreasing. Take $a > 0$, let T_a be the time of first return of $Y(t;\mathbf{0})$ to $\mathbf{0}$ after going into $(a,\infty)\times\mathbb{R}^{d-1}$ and define the measures m_λ, $\lambda \geq 0$, on $[0,\infty)\times\mathbb{R}^{d-1}$ by

$$\int f\,\mathrm{d}m_\lambda = E\left\{\int_0^{T_a} \mathrm{e}^{-\lambda t}f(Y(t;\mathbf{0}))\mathrm{d}t\right\}. \tag{4.7}$$

It is known that $c = 0$ if and only if $\inf\{X'(t) : t \geq 0\} = -\infty$ a.s. (e.g. see [7, p.363]). Therefore, under the condition $c = 0$, it is not so hard to prove that $T_a < \infty$ a.s. Since $\mathbb{Y}$ has the strong

Markov property (as can be easily verified), we have

$$\begin{aligned}\lambda^{-1}\int f\,\mathrm{d}\widehat{\nu}_\lambda &= E\left\{\int_0^\infty \mathrm{e}^{-\lambda t} f(Y(t;\mathbf{0}))\mathrm{d}t\right\}\\ &= E\left\{\int_0^{T_a} \mathrm{e}^{-\lambda t} f(Y(t;\mathbf{0}))\mathrm{d}t\right\} + E\left\{\int_{T_a}^\infty \mathrm{e}^{-\lambda t} f(Y(t;\mathbf{0}))\mathrm{d}t\right\}\\ &= \int f\,\mathrm{d}m_\lambda + E\{\mathrm{e}^{-\lambda T_a}\}E\left\{\int_0^\infty \mathrm{e}^{-\lambda t} f(Y(T_a+s;\mathbf{0}))\mathrm{d}t\right\},\end{aligned}$$

and hence

$$m_\lambda = \lambda^{-1}\left\{1 - E(\mathrm{e}^{-\lambda T_a})\right\}\widehat{\nu}_\lambda = \lambda^{-1}\left\{1 - E(\mathrm{e}^{-\lambda T_a})\right\}\widehat{c}_\lambda\widehat{\mu}_\lambda. \tag{4.8}$$

Since $m_\lambda \to m_0$ and $\widehat{\mu}_\lambda \to \widehat{\mu}$ as $\lambda \downarrow 0$, (4.8) implies that m_0 is a constant multiple of $\widehat{\mu}$. On the other hand, it is easy to see that m_0 is an invariant measure for $\mathbb{Y}$. Therefore $\widehat{\mu}$ is also an invariant measure for $\mathbb{Y}$. □

Next, we set $T_+ = \inf\{t > 0; X'(t) > 0\}$, $T_- = \inf\{t > 0; X'(t) < 0\}$ and consider the case $T_+ = T_- = 0$ a.s. By Itô's theory [4], there is a Poisson point process $\{p_t, t > 0\}$ of excursions around $\mathbf{0}$ attached to $\mathbb{Y}$. To be precise, in the case $c > 0$ $\{p_t, t > 0\}$ is an absorbed Poisson point process of Meyer [5]. Let W_0 (resp. W_1) be the space of right continuous excursions with left limits, starting at $\mathbf{0}$ and *ending up with jumps to* $\mathbf{0}$ *at finite terminal times* (resp. *never returning to* $\mathbf{0}$). Then, with an enlargement of the probability space if necessary, there is a Poisson point process $\{\widetilde{p}_t, t > 0\}$ on $W = W_0 \cup W_1$, an extension of $\{p_t, t > 0\}$, with the following properties (i), (ii), and (iii) (see [5]).

(i) Let $\mathbf{n}$ be the characteristic measure (excursion law) of $\{\widetilde{p}_t, t > 0\}$. Then

$$\mathbf{n}(W_0) = \infty \quad (\text{by } T_+ = T_- = 0 \text{ a.s.});$$
$$\mathbf{n}(W_1) = 0 \quad \text{or} \quad 0 < \mathbf{n}(W_1) < \infty \quad \text{according as } c = 0 \text{ or } c > 0.$$

(ii) Let T be the first time at which $\widetilde{p}_t$ takes a value in W_1. Then $T = \infty$ if $c = 0$, and T is exponentially distributed with mean $1/\mathbf{n}(W_1)$ if $c > 0$.

(iii) If $\mathcal{D}_p$ and $\mathcal{D}_{\widetilde{p}}$ denote the domains of definition of the point processes $\{p_t, t > 0\}$ and $\{\widetilde{p}_t, t > 0\}$, respectively, then $\mathcal{D}_p = \mathcal{D}_{\widetilde{p}} \cap (0, T] \subset (0, \infty)$ and $\{p_t, t > 0\}$ is the restriction of $\{\widetilde{p}_t, t > 0\}$ on $\mathcal{D}_p$.

For $f \in C_0(\mathbb{R}^d_+)$ and $w \in W$, we use the convention that $f(w(t)) = 0$ for any t larger than the terminal time of w, and set $F(w) = \int_0^\infty f(w(t))\mathrm{d}t$.

Proposition 10. *For any nonnegative measurable function f on W*

$$\int f\,\mathrm{d}\widehat{\mu} = const.\int_W F(w)\mathbf{n}(\mathrm{d}w), \tag{4.9}$$

namely,

$$\widehat{\mu} = const.\int_0^\infty e_t\,\mathrm{d}t, \tag{4.10}$$

where $\{e_t, t > 0\}$ is the entrance law associated with $\mathbf{n}$.

Proof. The method is to imitate that of [3, Theorem 3.1, Remark 4.2], The assumption $T_+ = T_- = 0$ a.s. implies $\int_0^\infty P\{Y(t;\mathbf{0}) = 0\}\mathrm{d}t = \int_0^\infty P\{X'(t) - N(t) = 0\}\mathrm{d}t = 0$, and so in the case $c > 0$ we have, from (1.13)

$$
\begin{aligned}
\int f\,\mathrm{d}\widehat{\mu} &= cE\left\{\int_0^\infty f(Y(t;\mathbf{0}))\mathrm{d}t\right\} \\
&= cE\left\{\sum_{0<s<T}\int_0^{\zeta(p_s)} f(p_s(t))\mathrm{d}t + \int_0^{\zeta(p_T)} f(p_T(t))\mathrm{d}t\right\} \\
&= cE\left\{\sum_{0<s<T} F(p_s) + F(p_T)\right\} \\
&= cE\{T\}\int_{W_0} F(w)\mathbf{n}(\mathrm{d}w) + \mathbf{n}(W_1)^{-1}\int_{W_1} F(w)\mathrm{d}\mathbf{n}(\mathrm{d}w) \\
&= c\mathbf{n}(W_1)^{-1}\int_W F(w)\mathbf{n}(\mathrm{d}w),
\end{aligned}
$$

proving (4.9) in the case $c > 0$. Next, consider the case $c = 0$. We use the same notation as in the proof of (iii) of Theorem 6. In the course of the proof of (iii) of Theorem 6, we have seen that $\widehat{\mu}$ is a constant multiple of m_0 defined by (4.7) with $\lambda = 0$. Therefore it is enough to compute $\int f\,\mathrm{d}m_0$. From the expression (4.7) with $\lambda = 0$, we have

$$
\int f\,\mathrm{d}m_0 = E\left\{\int_0^{T_a} f(Y(t;\mathbf{0}))\mathrm{d}t\right\} = E\left\{\sum_{0<s<S} F(p_s) + F(p_S)\right\},
$$

where S is the first time of $s \in \mathcal{D}_p$ for which p_s is in $W_a^+ = \{w : w(t) > a \text{ for some } t\}$. Therefore

$$
\begin{aligned}
\int f\,\mathrm{d}m_0 &= E\{S\}\int_{(W_a^+)^c} F(w)\mathbf{n}(\mathrm{d}w) + \mathbf{n}(W_a^+)^{-1}\int_{W_a^+} F(w)\mathbf{n}(\mathrm{d}w) \\
&= \mathbf{n}(W_a^+)^{-1}\int_W F(w)\mathbf{n}(\mathrm{d}w),
\end{aligned}
$$

which implies (4.9). □

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